Journal of Applied Mathematics and Mechanics 72 (2008) 306–311

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

The construction of a model of the growth of a defect for finite strains: Non-local criteria☆

V.A. Levin

Moscow, Russia

ARTICLE INFO

Article history:
Received 10 April 2006

ABSTRACT

A version of a model of the growth of a defect (an inclusion or cracks of non-zero opening) is presented. When constructing the model, the axiomatic of the mechanics of a deformable solid is used, as well as the idea of a zone of pre-existing imperfection as part of the body, in which, due to the action of external loads, applied to the body, a change in the properties of the material occurs. The apparatus of the theory of multiple superposition of large deformations and certain strength criteria are used.

1. Formulation of the problem within the framework of the mechanics of a deformable solid

When formulating strength problems within the framework of the mechanics of a deformable solid, the hypothesis of continuity must be taken into account. Hence, a description of the generation and growth of a defect, for example, cracks of non-zero opening, is only possible on the assumption that the crack either existed before the loading began, or was introduced into the body during the loading. For finite deformations, when describing the growth of a defect one must take into account the redistribution of finite deformations in the body.[1–5]

The instant when the growth of an existing crack begins is determined by the exceeding of the corresponding criterial quantity. In this case, to determine the stress-strain state of the body, the parameters of which occur in the theoretical part of the criterion, it is necessary to know the shape of the boundary surface of the crack either in the unloaded state of the body (in terms of the non-linear theory of elasticity – the initial state), or at the instant the criterion is satisfied (the actual one or the final one). In a number of models of the "forced" growth of a crack, the addition (removal) of external forces, for example, "cohesive forces",[6–8] which act on the surface or part of the surface of the crack, is used. Note that, when solving the problem of determining the stress-strain state of a body one must take into account the instant when these forces are applied, and, of course, the form of the boundary surface to which they are applied. For example, the application of these forces is possible either from the instant when all the external loads are applied, or from a definite instant, specified by the investigator (if it is assumed that the strains are not small). In this case we have the problem of the stage-by-stage loading of the body.[2,3,9,10]

It is assumed in the proposed model that the growth of a defect (crack) or its initiation occurs stage-by-stage. The idea of a pre-existing imperfection zone as part (or parts) of the body (zone), where the properties of the body material are changed due to the action of external loads applied to the body is used. The boundary of the pre-existing imperfection zone is found from the condition for satisfying the strength criterion. Apparently, in this case, the use of a non-local strength criterion, which takes into account the fact that the fracture and, of course, the change in the properties of the pre-existing imperfection zone, does not occur in a section and does not occur instantaneously (for a viscoelastic material) is the most acceptable one.

The physical meaning of this approach may be speculatively based on the fact that the body cannot be fractured by a mechanical field (external action) at a single point, since, neighbouring points must inevitably be involved in the fracture process, and this involvement is extended both in time and in space by virtue of the non-uniformity of the action of the mechanical field.[4,5] For example, for small deformations, it was shown in Ref. 11 that an attempt to average the stress intensity factor even along the whole length of the contour of a semi-elliptic surface crack gave satisfactory agreement with experiment.

☆ *Prikl. Mat. Mekh.* Vol. 72, No. 3, pp. 445–452, 2008.

E-mail address: v.a.levin@mail.ru.

0021-8928/$ – see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.jappmathmech.2008.07.010

2. A version of the construction of the model

For clarity, we will consider, as an example, a body of non-linear elastic material, capable of undergoing finite deformations.

At the first stage of the construction and use of the model we propose to describe the change in the properties of the body material in the pre-existing imperfection zone both by a change in the constants, which occur in the constitutive relations, without a change in the form of the constitutive relations, and by a change in the form of the constitutive relations. At the second stage, it is natural to introduce into consideration micropores and microinclusions which successively or simultaneously occur in the pre-existing imperfection zone, and to replace the material of the pre-existing imperfection zone by a change in its properties that is effective at each stage. This is important, for example, for elastomers in which, as was pointed out in Refs. 12–14, crystallization occurs, brought about by stretching. In the case of finite deformations, we can use the approach proposed previously in Refs. 3,15–18, to determine the properties of the effective material.

To solve the problem of the growth and change in the properties of the pre-existing imperfection zone for small deformations, we can use the apparatus developed to solve problems of inclusions and to determine the properties of the effective material,[19–21] and, for practical calculations, existing industrial software packages. It should be noted that, in this case, the main limitation is the postulation of the possibility of determining the stress-strain state of the body when there is total external action on the body by adding the parameters of the stress-strain state of the body for each action on it.

It is more difficult to solve the problem when it is assumed that the formation of the pre-existing imperfection zone is accompanied by a redistribution of finite deformations[22] (at least in the region of the pre-existing imperfection zone), which, probably, reflects the actual situation more exactly. In this case it is necessary to use the apparatus of the theory of repeated superposition of large deformations and to introduce the idea of a loading program. When using the idea of a loading program, it is assumed, in particular, that the formation of the pre-existing imperfection zone (or the growth of a cavity) occurs at the instant when a certain criterion is satisfied, rather than after the external forces reach specified values (or the application of all the loads to the body), i.e., before the loading program is completed. The loading program and the process (evolution) of the growth and development of the pre-existing imperfection zone (the degradation of the material, including its disappearance – splitting off) are interrelated.

The mechanical formulation of the problem when using a non-local criterion can be as follows.[3–5] Suppose a region has occurred in the body during the loading process, where the non-local criterion is exceeded. We mentally remove this region, and its action on the remaining part is replaced, by the principle of releasability from couplings, by forces distributed over this surface; this action does not change the stress-strain state of the remaining part of the body. We then fill the cavity formed by the removal of part of the body with an elastic material with different properties.

It is assumed that the shape of the inclusion obtained (which models the pre-existing imperfection zone) coincides with the shape of the removed part of the body, and forces are applied to it over the surface.[5] Further, the forces acting both on the boundary of the body, formed by the removal of part of it, and over the boundary of the pre-existing imperfection zone are reduced to zero quasistatically (for example, isothermally) without causing any dynamic affects in the body. This gives rise to additional finite deformations and stresses (at least in the neighbourhood of the inclusion), which are superimposed on the large deformations and stresses already existing in the body and in the pre-existing imperfection zone. According to the terminology of the theory of repeated superposition of large deformations, we can state that the body transfers to the next state.

We then check that the criterion is satisfied. If a new region occurs where the criterion is exceeded, the above-mentioned procedure of the formation of the pre-existing imperfection zone is repeated.

The formulation of the problem may be complicated, assuming that the pre-existing imperfection zone was formed at some stage when the loading program was being carried out. When the loading program is extended we must take into account the possibility that the pre-existing imperfection zone is formed with new properties (for example, inside the already existing pre-existing imperfection zone). One of these changes in the properties of the material may be a change in its density.

It is clear that such a process can be extended to the formation of a crack, which is modelled at a subsequent stage by the formation of a cavity, by removing part of the body. We mean by the removal of part of the body, for example, the “splitting off” of one or other part or a change in the properties of the “removed” part in such a way that it does not interact with the remaining part of the body.[5–22]

At the second stage of the construction of the model (refinement of the property of the material in the pre-existing imperfection zone) we take into account the fact that the initiation of a crack in the pre-existing imperfection zone is accompanied by complex physical-chemical processes, which lead, at the initial stage of the initiation of the crack, to the occurrence of differently (randomly) oriented microcracks and microinclusions, which then merge with the macrocrack.

In this case we can distinguish two approaches.

The first approach, which is included in the analysis of the mutual influence and interaction of these microcracks and microinclusions, enables us to take into account the considerable difference of the microstresses from their average values in the pre-existing imperfection zone.[23] In this case, by using the relations of the theory of repeated superposition of large deformations, we solve the problem of a body with stress concentrators, which arise in it successively when loading.[2–5,22] Note that, in this approach, it is possible to solve the problem for a system of simultaneously or successively occurring microcracks and their subsequent merging with the macrocrack using, for example, the model of the viscous growth of a crack; problems of this kind were considered in Ref. 5. This approach, in our opinion, is difficult from the calculation point of view and is not always rational, since the investigator specifies the position of the microcracks.

The second approach consists of replacing the material in the pre-existing imperfection zone, containing the microcracks and microdamage, by an effective material as in the procedure proposed earlier.[3,15–18] The problem of a body with a pre-existing imperfection zone is then solved, in which it is desirable to take into account the random nature of the distances between the microcracks and the microdamage and their dimensions.

We will present the relations of the theory of the repeated superposition of large deformations, which enables us to describe the proposed models. These relations are used when solving specific problems.

For simplicity we will derive these relations for a non-linear elastic body. We will assume that the conditions for the corresponding displacements and stresses to be equal on the boundary of the pre-existing imperfection zone are satisfied.

The system of equilibrium equations and boundary conditions have the form[3]

$$\overset{n}{\nabla} \cdot \overset{n}{\Sigma}_{0,m} - \overset{n}{\Sigma}_{0,m} \cdot \overset{n}{\nabla} \ln(1+\Delta_{0,m}) + \overset{n}{\Sigma}_{0,m} \cdot\cdot (\overset{n}{\nabla}\Psi_{n,m}) \cdot \Psi_{n,m}^{-1} - (\overset{n}{\nabla} \cdot \Psi_{n,m}^{*-1}) \cdot \Psi_{n,m}^{*} \cdot \overset{n}{\Sigma}_{0,m} = 0 \tag{2.1}$$

$$\overset{n}{\mathbf{N}} \cdot \overset{n}{\Sigma}_{0,m} = \overset{n}{\mathbf{P}}{}_N^{(m)}, \quad \overset{n}{\mathbf{P}}{}_N^{(m)} = (1+\Delta_{0,m}) \frac{|d\overset{n}{\tau}|}{|d\overset{m}{\tau}|} \overset{n}{\mathbf{P}}{}_N^{(n)} \cdot \Psi_{m,n}^{-1} \tag{2.2}$$

Here

$$\overset{n}{\Sigma}_{0,m} = (1+\Delta_{0,m}) \Psi_{n,m}^{*} \cdot \sigma_{0,m} \cdot \Psi_{n,m}, \quad \Psi_{n,m} = \overset{m}{\mathbf{e}}{}^{i} \overset{n}{\mathbf{e}}_i, \quad \overset{n}{\mathbf{e}}_i = \frac{\partial \overset{n}{\mathbf{r}}}{\partial \xi^i} \tag{2.3}$$

$\sigma_{0,m}$ is the tensor of the true total stresses for the m-th state ($\sigma_{0,1}$ is the Cauchy tensor), $\overset{n}{\sum}_{0,m}$ is the tensor of the generalized total stresses for the m-th state, referred to the coordinate basis of the n-th state, and $\Delta_{0,m}$ is the relative change in the volume when the body changes from the initial state to the m-th state. $\Psi_{m,n} = \Psi_{0,m}^{-1} \cdot \Psi_{0,n}$ is the corresponding deformation gradient,[3,24,25] $\overset{n}{\mathbf{r}}$ is the radius vector of a point (a particle of the body) in the n-th state, ξ^i are the Lagrangian coordinates of the particle, and $\overset{n}{\mathbf{r}} - \overset{n-1}{\mathbf{r}} = \mathbf{u}_n$, $\mathbf{u}_n$ is the vector of the displacements, characterizing the displacement from the previous (n–1)-th state to the subsequent n-th state, where

$$\Psi_{q,p} = I + \sum_{n=q+1}^{p} \overset{q}{\nabla} \mathbf{u}_n = \left(I - \sum_{n=q+1}^{p} \overset{p}{\nabla} \mathbf{u}_n\right)^{-1}, \quad \overset{p}{\nabla} = \overset{p}{\mathbf{e}}{}^{i} \frac{\partial}{\partial \xi^i} \tag{2.4}$$

The sign over a symbol denotes the number of the state in which the given quantity (apart from $\overset{n}{\mathbf{r}}$ and $\overset{n}{\mathbf{e}}$) is specified or to which it relates and $\overset{n}{\mathbf{P}}{}_N^{(n)}$ is the vector of the actual stresses on the area $d\overset{n}{\tau}$. The conditions on the boundary between the body (the matrix) and the pre-existing imperfection zone (inclusion) have the form (and can be converted[3] to the coordinate basis of the corresponding state)

$$\overset{n}{\mathbf{N}}_n \cdot \sigma_{0,n}\big|_{\times} = \overset{n}{\mathbf{N}}_n \cdot \sigma_{0,n}\big|_{\times\times}, \quad \mathbf{u}_n\big|_{\times} = \mathbf{u}_n\big|_{\times\times} \tag{2.5}$$

where the cross indicates that the corresponding quantity is taken on the boundary of the matrix in the n-th state, while the double cross indicates that the corresponding quantity is taken on the boundary of the pre-existing imperfection zone in the n-th state.

When calculating the model examples we will use the model of a compressible Murnaghan material[25]

$$\overset{0}{\Sigma}_{0,n} = \frac{1}{2}(\Psi_{0,n} \cdot \Psi_{0,n}^{*} - I) = \lambda(\overset{0}{E}_{0,n} \cdot\cdot I)I + 2G\overset{0}{E}_{0,n} + 3C_3(\overset{0}{E}_{0,n} \cdot\cdot I)^2 I + C_4(\overset{0}{E}{}^2_{0,n} \cdot\cdot I)I +$$
$$+ 2C_4(\overset{0}{E}_{0,n} \cdot\cdot I)\overset{0}{E}_{0,n} + 3C_5(\overset{0}{E}_{0,n})^2 \tag{2.6}$$

The asterisk denotes transposition.

In the general case[3]

$$\overset{k}{E}_{m,n} = \frac{1}{2}(\Psi_{k,n} \cdot \Psi_{k,n}^{*} - \Psi_{k,m} \cdot \Psi_{k,m}^{*}) \tag{2.7}$$

$\overset{k}{E}_{m,n}$ is the strain tensor, which describes the change in the strains on changing from the m-th to the n-th state and relates to the coordinate basis of the n-th state (note that when $m > n$ in (2.7) the order of the subscripts is changed) and $\overset{1}{E}_{0,1}$ is the Almansi tensor.

It can be seen from relations (2.1)–(2.7) that if $m > n$, the tensor $\overset{m}{\sum}_{0,n}$ depends not only on the strain tensor $\overset{m}{E}_{0,m}$ but also on the strain tensor $\overset{m}{E}_{n,m}$. This considerably complicates the solutions of the problems when they are formulated in the space of the m-th state, since, in particular, the system of equilibrium equations (in the general case of m vector equations of the overall number of equations) does not decompose into individual equations (and therefore, as already pointed out, the use of standard software packages, designed to solve problems of elasticity and viscoelasticity, is practically impossible[3,5]). But this formulation is often necessary, for example, when the boundary conditions are specified in the m-th state.

When determining the pre-existing imperfection zone one can use different criteria. Apparently, the most acceptable are non-local criteria, which take into account that a change in the properties of the material in the pre-existing imperfection zone and subsequent damage in it occur in a finite region and not instantaneously.[4,5] Hence, the non-local criterion, in addition to a criterial quantity (we will call it V_k, and usually this is a combination of components of the Cauchy or Almansi tensor) should contain a parameter characterizing the size of the region being analysed.

A non-local strength criterion for the finite deformations of bodies of elastic and viscoelastic materials was proposed in Ref. 4, which is useful when analysing the viscous growth of a crack. We note that fracture criteria in the form of averaging of the criterial quantity both over space and time (but not for finite deformations and not for bodies of viscoelastic material) are known (see, for example, Refs. 8,26,27), while we must assume that the Wieghardt criterion[28] was probably the first such criterion.

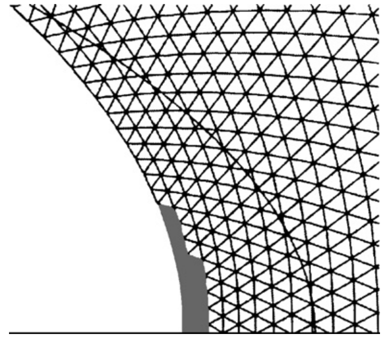

Fig. 1.

Below we describe the simplest (for clarity) non-local criterion for determining the pre-existing imperfection zone and the development of fracture.[29] The criterion is based on the use of a local criterion of the form $V = V_k$ to find the pre-existing imperfection zone and on the postulation for such an excess (α, %) "at a dangerous point" of this quantity that a pre-existing imperfection zone is instantaneously formed. This criterion is satisfied on the line (isoline) enveloping the pre-existing imperfection zone, rather than at a point. Here it is assumed that, in the region enveloped by the isoline, the level of the criterial quantity V exceeds the limit of the quantity V_k but by no more than α, %. On this isoline the strength condition is satisfied according to the local criterion $V = V_k$ (in the region enclosed by isoline $V_k \leq V \leq V_k\,(1 + \alpha/100)$).

In the region bounded by this isoline, a degree of pre-existing imperfection is postulated, for example, in the form of pre-specified material porosity (reflecting its damage) or the occurrence of microinclusions. The material in the pre-existing imperfection zone acquires other (effective) elastic properties. The occurrence of a zone with new properties leads to a redistribution of the stresses and strains in the body, leading to a change in the form of the pre-existing imperfection zone.

We will further change (by a certain pre-specified amount Δp) the external load (or we apply a new one) according to a loading program. We find, correspondingly, the stress-strain state. As above, we verify that the non-local criterion is satisfied in the material (with a determination of the new boundary of the pre-existing imperfection zone). Inside the pre-existing imperfection zone, using a similar non-local criterion (if it is satisfied) we find the pre-existing imperfection zone of the next level, in which the properties of the material of the zone vary, for example, we increase the porosity. Naturally, the occurrence of a pre-existing imperfection zone with new properties leads to a redistribution of the stresses and strains in the body.

Continuing the loading and using the scheme described above, we determine the new pre-existing imperfection zone, for example, with an ever increasing porosity, until the porosity in the last pre-existing imperfection zone (at the last stage) becomes "limiting". The fact that a limit porosity is reached indicates that the material has "disappeared" (has "split off") in the last pre-existing imperfection zone. This usually occurs at the boundary of the defect.

One can increase the external load even further, thereby analysing the development of the pre-existing imperfection zone and the progressing splitting off of the material.

Note that it is assumed in this model that the pre-existing imperfection zone is formed and changes instantaneously, and possible refinements can only be related to their "physical", rather than "theoretical" development.

3. Results

We will consider, as a model problem, which illustrates this approach, the plane problem of the growth of an elliptic crack under a uniaxial initial tension ($\sigma_{11}^{\infty} = 0, \sigma_{22}^{\infty} = p$). The calculations were carried out using a specialized "application" software package based on the finite element method. For specific calculations we used a finite body (a square) with a side 20 times greater than the characteristic size of the crack.

It was assumed that the material of the body is non-linear elastic and is described by the Murnaghan model ($\lambda/G = 2.097$, $C_3/G = -0.0689$, $C_4/G = -0.3746$, $C_5/G = 0.3371$).[25] The elliptic crack (with a ratio of the semiaxes of 7) existed in the unloaded body. It is assumed that two pre-existing imperfection zones are formed. The first occurs when $V_k/G = 0.8407$ ($\alpha = 0.05$, $V_k = \sigma_1$). This criterial value was achieved for a load of $p/G = 0.05$. Expansion of the micropores occurs in the pre-existing imperfection zone (porosity 0.06). The effective properties of the material were determined using the well-known method described in Refs. 3,15–18, and enabled us to determine the effective properties of the material for finite deformations and their redistribution. Inside the first pre-existing imperfection zone, when the loading program was carried out, a second pre-existing imperfection zone is formed when $V_k/G = 1.167$ for a load $p/G = 0.0751$ ($\alpha = 0.05$). It is assumed that additional expansion of the micropores occurs in the second pre-existing imperfection zone (porosity 0.09). Evolution of the first pre-existing imperfection zone naturally also occurs. Further, when the loading program is carried out, splitting off occurs when $V_k/G = 1.456$ (for a load $p/G = 0.1125$).

In Figs. 1–3 we show (on a finite-element grid) the shapes of the contour of the crack (defect) and the pre-existing imperfection zone at the instant when the first and second pre-existing imperfection zones are formed and after splitting off; the pre-existing imperfection zones are shown grey, and the second zone is shown darker; the continuous curve shows the shape of the contour of the crack in the initial state. In Fig. 4 we show the shape of the contour of the crack, ignoring the formation of the pre-existing imperfection zone after the loading program is completed. The solution without taking the pre-existing imperfection zone and the "forced" growth of the crack into account shows that the stress level at the tip of the defect is 17–25% greater at different stages of the loading.

Hence, we have proposed a model, within the framework of the mechanics of a deformable solid, for the case of finite deformations, which enables the stage-by-stage degradation of the material in the pre-existing imperfection zone to be taken into account. The use of the

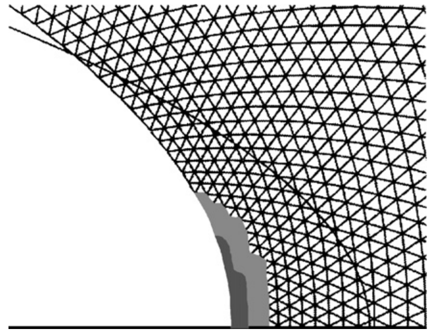

Fig. 2.

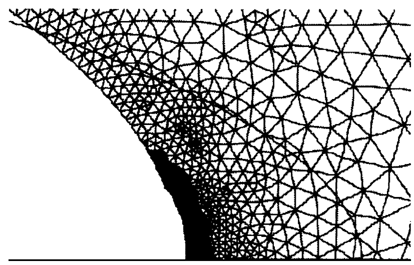

Fig. 3.

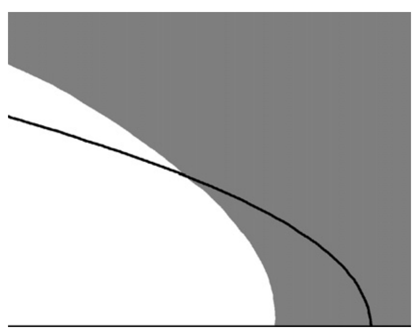

Fig. 4.

theory of repeated superposition of large deformations has enabled us to take into account the redistribution of the finite deformations, due both to a change in the properties of the material in the pre-existing imperfection zone and due to the loading program. In our opinion, this approach enables the results and approaches of research, in which the damage (for example, Refs. 30–33) for finite deformations and their redistribution are taken into account, including for incompressible materials,[25] to be used. One can similarly use "averaging" criteria,[3,5] by specifying (but not determining) the shape of the pre-existing imperfection zone in the criterion.

The calculations in the problem were carried out together with A. V. Vershinin.

Acknowledgements

I wish to thank Ye. M. Morozov for useful discussions. This research was financed by the Russian Foundation for Basic Research (06-01-00682).

References

1. Cherepanov GP. *Brittle Fracture Mechanics*. Moscow: Nauka; 1974.
2. Levin VA. Theory of repeated superposition of large deformations. Elastic and viscoelastic bodies. *Intern J Solids and Structures* 1998;**35**(20):2585–600.
3. Levin VA. *Repeated Superposition of Large Deformations in Elastic and Viscoelastic Solids*. Moscow: Nauka, Fizmatlit; 1999.
4. Levin VA, Morozov YeM. A non-local fracture criterion. Finite deformations. *Dokl Ross Akad Nauk* 2002;**386**(1):46–7.
5. Levin VA, Morozov YeM, Matvienko YuG. *Selected Non-linear Problems of Fracture Mechanics*. Moscow: Fizmatlit; 2004.
6. Barenblatt GI. The mathematical theory of the equilibrium of brittle fracture cracks. *Zh Prikl Mekh Tekh Fiz* 1961;**4**:3–56.
7. Leonov MYa. *Deformation and Fracture Mechanics*. Frunze: Il'm; 1981.
8. Panasyuk VV (Ed.), Fracture Mechanics and Strength of Materials. A Reference Textbook. Kiev: Naukova Dumka; Vol. 1, 1988; Vol. 2, 1988; Vol. 3, 1988, Vol. 4, 1990.
9. Levin VA. A "physical cut" introduced into a preloaded elastic solid. Finite deformations. *Dokl Ross Akad Nauk* 2001;**381**(2):196–8.
10. Levin VA. Stress concentration about an opening that is circular at the instant of formation in a solid of viscoelastic material. *Dokl Akad Nauk SSSR* 1988;**299**(5):1079–82.
11. Morozov YeM, Kostenko PV. The method of sections for calculating natural articles with cracks. *Zavod Lab Diagnos Materialov* 1999;**65**(7):31–4.
12. Shur AM. *High Molecular Weight Compounds*. Moscow: Vyssh Shkola; 1981.

13. Bukhina MF. *Technical Physics of Elastomers*. Moscow: Khimiya; 1984.
14. Bukhina MF. *Crystallization of Rubber and Vulcanized Rubber*. Khimiya: Moscow; 1973.
15. Levin V, Kachanov M, Lokhin V, Zingerman K. Effective response of porous materials undergoing large deformations. *19th Intern Congr Theor and Appl Mech: Abstr Kyoto* 1996:262.
16. Levin VA, Lokhin VV, Zingerman KM. Effective elastic properties of porous materials with randomly disposed pores. Finited deformation. *Trans ASME J Appl Mech* 2000;**67**(4):667–70.
17. Levin VA, Zingerman KM. Effective constitutive equations for porous elastic materials at finite strains and superimposed finite strains. *Trans ASME J Appl Mech* 2003;**70**(6):809–16.
18. Levin VA, Zingerman KM. The construction of effective constitutive relations for porous elastic materials for finite deformations and their superposition. *Dokl Ross Akad Nauk* 2002;**382**(4):482–7.
19. Kachanov M, Sevostianov I. On quantitative characterization of microstructures and effective properties. *Intern J Solids and Structures* 2005;**42**(2):309–36.
20. Kachanov M, Tsukrov I, Shafiro B. Effective moduli of solids with cavities of various shapes. *Appl Mech Rev* 1994;**47**(1):151–74.
21. Pobedrya BYE. *Mechanics of Composite Materials*. Moscow: Izd MGU; 1984.
22. Levin VA. Modelling of the growth of a defect under finite deformations. *Vestnik MGU Ser 1 Mat Mekh* 2006;**3**:38–41.
23. Levin VA, Zingerman KM. The effect of small defects on the stress concentration around a hole. *Dokl Ross Akad Nauk* 2002;**386**(4):478–81.
24. Sedov LI. *Introduction to Continuum Mechanics*. Moscow: Fizmatgiz; 1962.
25. Lur'e AI. *Non-linear Theory of Elasticity*. Moscow: Nauka; 1980.
26. Morozov YeM. The correspondence between the energy fracture criterion and mathematical modelling of deformation phenomena at the tip of cut-cracks. *Prikl Mat Mekh* 1970;**34**(4):768–76.
27. Morozov NF, Petrov YuV. The concept of structural time and the theory of the dynamic fracture of brittle materials. *Dokl Ross Akad Nauk* 1992;**324**(5):964–7.
28. Wieghardt K. Über spalter und zerressen elastischer Korper. Zeitschr Math Phys 1907. Bd. 55. H. 1/2. S. 60-103.
29. Levin VA, Morozov YeM. Non-local criteria for determining the pre-existing imperfection zone when describing the growth of a defect for finite deformations. *Dokl Ross Akad Nauk* 2007;**415**(1):52–4.
30. Astaf'ev VI, Radayev YuN, Stepanova LV. *Non-linear Fracture Mechanics*. Samara: Izd Samar Univ; 2001.
31. Kachanov LM. *Principles of Fracture Mechanics*. Moscow: Nauka; 1974.
32. Rabotnov YuN. *The mechanism of persistent fracture In Problems of the Strength of Materials and Structures*. Moscow: Izd Akad Nauk SSSR; 1959. pp. 5–7.
33. Kachanov LM. *Introduction to Continuum Damage Mechanics*. Dordrecht; Boston: Nijhoff; 1986. p. 135.

Translated by R.C.G.